

Superfields, nilpotent superfields and superschemes

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Abstract

We define superfields using a functorial formalism that explains some paradoxical properties they are supposed to have. We also investigate some algebraic constraints on them giving rise to superschemes that, generically, are not regular, that is, they do not define a standard supermanifold.

1 Introduction

This work arises in the spirit of conciliating two apparently different points of view to see super objects. Mathematicians understand supervarieties and super manifolds by means of sheaves of superalgebras that properly generalize similar definitions of algebraic geometry. Although consistent and very successful, this approach seems somewhat removed from the language that physicists use.

The definition of an ‘even’ or ‘bosonic’ fields offers no difficulty: they are functions on spacetime valued in a finite dimensional manifold, sections of a vector bundle, connections over it... All these objects have a precise geometrical meaning and offer no ambiguity both, in the physics use and in their mathematical formulation.

An ‘odd’ or ‘fermionic’ field is more difficult to interpret. In physics, it is stated vaguely that it is an ‘odd function on spacetime’, that is, in the simplest case, fermionic fields are functions with values in some super vector space (in its odd part) or superalgebra. However, physicists use properties of these fields that cannot be reconciled with this simple point of view. For example, it is assumed without further explanation that the ‘values’ of the fermionic field in two different points of space time can be multiplied and the result is, generically, different from zero, which cannot be the case in the naive approach exposed before. It is enough to consider a one dimensional odd field to see that this description is not adequate.

There have been several attempts of properly defining what is an odd field and also a superfield. In one way or another, all of them would make use of infinite Grassmann variables. Sometimes it has been proposed to attach a copy of a Grassmann algebra at every point of spacetime, which seems to us a very artificial way of looking at the problem, mainly because, after all, one has to be able to take derivatives of the fields and then compare the fibers at different points. Needless to say, this hyperabundance of odd variables has to disappear at the end, since they are spurious variables that are called only to reproduce certain properties of the odd fields.

In a more modern approach to supergeometry ([1, 2, 3, 4]) the paradox can be solved in an elegant way. There are extra odd variables, but these are spurious because they have a *functorial* behavior: we will see in detail how this works. The only old reference where we were able to find an allusion to functoriality is Ref. [5].

Moreover, in some supersymmetric theories (we will see examples in sec-

tions 4 and 5) some constraints are imposed on superfields that defy naive descriptions: we will see that the use of *superschemes* is necessary to provide an adequate mathematical framework for them.

The paper is organized as follows: In Section 2 we describe a theorem by Deligne and Morgan (following Bernstein), the *even rules principle*, that will be the basis of our considerations. In Section 3 we first define what is a scalar superfield and relate it to the concept of superspace. In sections 4 and 5 we describe in detail some non linear constraints (nilpotent conditions) that can be imposed on scalar superfields. These are constraints that have been used in some supergravity inspired cosmological models. We will see how the use of superschemes clarifies the interpretation of such constraints and allows to study their behaviour under supersymmetry. Finally, in Section 6 we comment on the interpretation of fermionic observables in the classical and quantum realms.

In the text we have tried to introduce the basic notions of algebraic geometry that are required to understand the generalization to the super setting. Some more basic concepts, as the definition of sheaf, etc are given in the Appendix A. We have tried to give a consistent account of these concepts as a guide for the reader, but this paper is not a suitable place to learn in depth algebraic geometry, for which many good textbooks exist (particularly useful for us has been Ref. [6]). For the super setting, introductory references are Refs. [1, 2, 4] and a more detailed monograph is Ref. [3]. Physics conventions regarding spinor notation and supersymmetry transformations are given in Appendices B and C.

Some mathematical notation:

1. Let A be an algebra (not necessarily commutative) and let $a_1, a_2, \dots, a_n \in A$. We denote by $\langle a_1, a_2, \dots, a_n \rangle$ the two sided ideal generated by those elements:

$$\langle a_1, a_2, \dots, a_n \rangle := \left\{ \sum_{i=1}^n b_i a_i c_i, \quad b_i, c_i \in A \right\}.$$

2. We denote as $k[x, y, \dots, \theta, \psi, \dots]$ the commutative superalgebra freely generated by the even (commuting) variables x, y, \dots and the odd (anticommuting) variables θ, ψ, \dots . In general, even variables will be denoted with Latin letters and odd variables with Greek ones.

2 The even rules principle

In this section we give some mathematical definitions and results that will be needed in order to understand what is a superfield in the functorial approach. The main result is at the end of the section, the *even rules principle*, due to Deligne and Morgan [1].

We set $k = \mathbb{R}, \mathbb{C}$. The algebras and superalgebras that we consider here are k -algebras, unless otherwise stated, and have always unit.

We will denote by (svector spaces) the category of super vector spaces and by (csalgebras) the category of commutative¹ superalgebras with unit, both over k .

Let \mathcal{A} be a commutative superalgebra and $M_{\mathcal{A}}$ a left \mathcal{A} -module, that is, $M_{\mathcal{A}}$ is a super vector space with a morphism of super vector spaces

$$\begin{aligned}\mathcal{A} \otimes M_{\mathcal{A}} &\longrightarrow M_{\mathcal{A}} \\ a \otimes m &\longrightarrow a \cdot m,\end{aligned}$$

satisfying

$$a \cdot (b \cdot m) = (ab) \cdot m, \quad a, b \in \mathcal{A}, \quad m \in M_{\mathcal{A}}.$$

Left \mathcal{A} -modules for commutative superalgebras are also right \mathcal{A} -modules

$$\begin{aligned}M_{\mathcal{A}} \otimes \mathcal{A} &\longrightarrow M_{\mathcal{A}} \\ m \otimes a &\longrightarrow m \cdot a := (-1)^{p(m)p(a)} a \cdot m,\end{aligned}$$

where $p(m)$ and $p(a)$ are the parities of m and a respectively. This satisfies

$$(m \cdot b) \cdot a = m \cdot (b \cdot a), \quad a, b \in \mathcal{A}, \quad m \in M_{\mathcal{A}},$$

We will just call them modules.

Let V be an object in (svector spaces) and \mathcal{B} an object in (csalgebras). We denote as $V(\mathcal{B}) := \mathcal{B} \otimes V$ the extension of the scalars of V by \mathcal{B} (see Definition A.1).

Let $h : \mathcal{B} \rightarrow \mathcal{B}'$ be a morphism of commutative superalgebras. Then $V(\mathcal{B}')$ is also a \mathcal{B} -module by further extending the scalars to \mathcal{B}' . There is a morphism of \mathcal{B} -modules

$$\begin{aligned}V(\mathcal{B}) &\xrightarrow{V(h)} V(\mathcal{B}') \\ b \otimes v &\longrightarrow h(b) \otimes v.\end{aligned}$$

¹We stick to the categorical notation [1]. A commutative superalgebra is sometimes called *supercommutative* in the physics literature.

This is well defined since h is a superalgebra morphism.

Let $V = V_0 + V_1$ and $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1$ be the splitting in even and odd parts of V and \mathcal{B} . We denote

$$\begin{aligned} V_0(\mathcal{B}) &:= (\mathcal{B} \otimes V)_0 = \mathcal{B}_0 \otimes V_0 + \mathcal{B}_1 \otimes V_1, \\ V_1(\mathcal{B}) &:= (\mathcal{B} \otimes V)_1 = \mathcal{B}_0 \otimes V_1 + \mathcal{B}_1 \otimes V_0. \end{aligned}$$

$V_0(\mathcal{B})$ and $V_1(\mathcal{B})$ are \mathcal{B}_0 -modules. Given a morphism of superalgebras $h : \mathcal{B} \rightarrow \mathcal{B}'$, then $V(h)$ is a morphism of \mathcal{B}_0 -modules

$$\begin{aligned} V_{0,1}(\mathcal{B}) &\xrightarrow{V(h)} V_{0,1}(\mathcal{B}') \\ b \otimes v &\longrightarrow h(b) \otimes v. \end{aligned}$$

Remark 2.1. It will be convenient in the following to express the definition of superalgebra in terms of commutative diagrams.

Let V be a super vector space. An associative superalgebra structure on V is given by a linear map, *the product*:

$$V \otimes V \xrightarrow{\pi} V$$

such that

$$p((\pi(u \otimes v))) = p(u) + p(v)$$

and satisfying the associativity property, that is, the diagram

$$\begin{array}{ccc} & V \otimes V & \\ p \oplus \Pi \nearrow & & \searrow p \\ V \otimes V \otimes V & & V \\ \Pi \oplus p \searrow & & \nearrow p \\ & V \otimes V & \end{array}$$

is commutative. On the other hand, we say that the superalgebra is commutative if, given the *flip map*,

$$\begin{aligned} V \otimes V &\xrightarrow{c_{V,V}} V \otimes V \\ v \otimes w &\longrightarrow (-1)^{p_v p_w} w \otimes v, \end{aligned}$$

the diagram

$$\begin{array}{ccc}
 & V \otimes V & \\
 \nearrow c_{V,V} & & \downarrow p \\
 V \otimes V & \xrightarrow{p} & V
 \end{array}$$

commutes. □

If V has a k -superalgebra structure, then $V(\mathcal{B})$ has a \mathcal{B}_0 -algebra structure naturally: Let $p : V \otimes V \rightarrow V$ be the product on V , then we have a product²

$$V(\mathcal{B}) \times V(\mathcal{B}) \xrightarrow{\pi_{\mathcal{B}}} V(\mathcal{B}) \quad (1)$$

given simply by

$$\pi_{\mathcal{B}}(b_1 \otimes v_1, b_2 \otimes v_2) = (-1)^{p(v_1)p(b_2)} b_1 b_2 \pi(v_1 \otimes v_2). \quad (2)$$

It is straightforward to check that the associativity and commutativity properties are satisfied.

Definition 2.2. Let V and W be two superspaces. We say that a family of morphisms

$$\{ f_{\mathcal{B}} : V(\mathcal{B}) \rightarrow W(\mathcal{B}), \quad \mathcal{B} \in (\text{c salgebras}) \}$$

is *functorial* in \mathcal{B} if given a superalgebra morphism

$$\mathcal{B} \xrightarrow{h} \mathcal{B}'$$

the diagram

$$\begin{array}{ccc}
 V(\mathcal{B}) & \xrightarrow{f_{\mathcal{B}}} & W(\mathcal{B}) \\
 V(h) \downarrow & & \downarrow W(h) \\
 V(\mathcal{B}') & \xrightarrow{f_{\mathcal{B}'}} & W(\mathcal{B}').
 \end{array}$$

commutes. □

²We use here the direct product and not the tensor product because the bilinearity is in \mathcal{B}_0 and not only in k .

In particular, if V has a superalgebra structure, it is not difficult to see that the family $\{\pi_{\mathcal{B}} : V(\mathcal{B}) \otimes V(\mathcal{B}) \rightarrow V(\mathcal{B})\}$ is functorial in \mathcal{B} . The same is true for the families of maps appearing in the associativity and commutativity diagrams for the algebras $V(\mathcal{B})$.

The following theorem of Deligne and Morgan (Ref.[1], page 56) will be key in making connection with the physics notion of superfield.

Theorem 2.3. *Even rules principle. Let $\{V_i\}_{i \in I}$, $I = 1, \dots, n$ be a family of super vector spaces, V another super vector space and $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ a commutative superalgebra. As before, we denote $V_{i0}(\mathcal{B}) = (\mathcal{B} \otimes V_i)_0$ and $V_0(\mathcal{B}) = (\mathcal{B} \otimes V)_0$.*

Any family of \mathcal{B}_0 -multilinear maps

$$V_{10}(\mathcal{B}) \times \dots \times V_{n0}(\mathcal{B}) \xrightarrow{f_{\mathcal{B}}} V_0(\mathcal{B})$$

which is functorial in \mathcal{B} comes from a unique morphism

$$V_1 \otimes \dots \otimes V_n \xrightarrow{f} V$$

as in (1), that is,

$$f_{\mathcal{B}}(b_1 \otimes v_1, b_2 \otimes v_2, \dots, b_n \otimes v_n) = (-1)^p b_1 \dots b_n f(v_1 \otimes \dots \otimes v_n),$$

where p is the number of pairs (i, j) with $i < j$ and v_i, v_j odd.

Proof. We will not prove the theorem here (see Ref. [1]), but it is instructive to see how the map f can be recovered from the family of maps $f_{\mathcal{B}}$. Let us consider the simple case of a family of maps

$$V(\mathcal{B}) \otimes V(\mathcal{B}) \xrightarrow{p_{\mathcal{B}}} V(\mathcal{B}),$$

then we have three possible cases:

1. v_1, v_2 even. Then we may take $b_1 = b_2 = 1$ (in an arbitrary algebra \mathcal{B}) and

$$p(v_1 \otimes v_2) := p_{\mathcal{B}}(v_1, v_2).$$

2. v_1 even, v_2 odd. Then we take for example $\mathcal{B} = \Lambda(\xi)$, $b_1 = 1$ and $b_2 = \xi$. The equality

$$p_{\mathcal{B}}(v_1, \xi v_2) = \xi p(v_1 \otimes v_2)$$

determines $p(v_1 \otimes v_2)$.

3. v_1 odd, v_2 odd. It is enough to consider $\mathcal{B} = \Lambda(\xi^1, \xi^2)$ and the equality

$$p_{\mathcal{B}}(\xi^1 v_1, \xi^2 v_2) = -\xi^1 \xi^2 p(v_1 \otimes v_2)$$

determines $p(v_1 \otimes v_2)$.

□

Remark 2.4. One way to give a structure of superalgebra to V is to give a \mathcal{B}_0 -algebra structure on $V_0(\mathcal{B})$ which varies functorially with \mathcal{B} . From the commutative diagrams in Remark 2.1, it is clear that the superalgebra will be associative or commutative if the algebra structures on the $V_0(\mathcal{B})$'s are so.

□

Example 2.5. *Toy model.* Let us consider the vector space over the reals with basis one even vector e and one odd vector θ .

$$V_{\mathbb{R}}^{1|1} = \text{span}\{e, \theta\},$$

and let us consider the functor (2). Then, one element of $V_{\mathbb{R},0}^{1|1}(\mathcal{B})$ will be of the form

$$\Phi_{\mathcal{B}} = b_0 \otimes e + b_1 \otimes \theta, \quad b_0 \in \mathcal{B}_0, \quad b_1 \in \mathcal{B}_1.$$

We can define a product on it as

$$\Psi_{\mathcal{B}} \bullet_{\mathcal{B}} \Phi_{\mathcal{B}} = b_0 a_0 \otimes e + (b_0 a_1 + a_1 b_0) \otimes \theta.$$

It is immediate to check the functorial property (Definition 2.2) of the whole family of products $\{\bullet_{\mathcal{B}}\}$: for a morphism $h : \mathcal{B} \rightarrow \mathcal{B}'$, we have the map (2)

$$V_{\mathbb{R},0}^{1|1}(\mathcal{B}) \xrightarrow{V_{\mathbb{R}}^{1|1}(h)} V_{\mathbb{R},0}^{1|1}(\mathcal{B}'),$$

that, for short and without risk of confusion, we will call simply h

$$\begin{aligned} h(\Psi_{\mathcal{B}} \bullet_{\mathcal{B}} \Phi_{\mathcal{B}}) &= h(b_0 a_0) \otimes e + h(b_0 a_1 + a_1 b_0) \otimes \theta = \\ &= h(b_0)h(a_0) \otimes e + (h(b_0)h(a_1) + h(a_1)h(b_0)) \otimes \theta = \\ &= h(\Phi_{\mathcal{B}}) \bullet_{\mathcal{B}'} h(\Psi_{\mathcal{B}}). \end{aligned}$$

One can also check the functoriality for the associativity and commutativity diagrams (Remark 2.1). The algebra structure defined in the superspace $V_{\mathbb{R}}^{1|1}$ converts it into the Grassmann algebra in one variable $\Lambda(\theta)$.

□

A pair of remarks appearing in Ref. [1] will be of use to us. We state them here without proof.

Remark 2.6. One can substitute super vector spaces over a field k by modules \mathcal{M} over a fixed, commutative superalgebra, say \mathcal{A} . We demand that \mathcal{B} is a commutative \mathcal{A} -superalgebra, so it is itself an \mathcal{A} -module and one can define the tensor product $\mathcal{M}(\mathcal{B}) = (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M})$ as in Definition A.1. As before, we have $\mathcal{M}(\mathcal{B}) = (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}) = \mathcal{M}_0(\mathcal{B}) + \mathcal{M}_1(\mathcal{B})$, which are all \mathcal{B}_0 -modules. \square

Remark 2.7. One can obtain the same result if, instead of considering arbitrary commutative superalgebras, we restrict to consider Grassmann algebras $\mathcal{B} = \wedge[\xi^1, \dots, \xi^n]$, for short \wedge^n , for n arbitrarily large, but finite.

In applications, it will be useful to consider \wedge^n instead of the full category (c salgebras). The object $(\wedge^n \otimes V)_0$ (with n not specified) is sometimes called the *Grassmann envelope* of the vector space V . \square

3 Superspaces and scalar superfields

In this section we want to progress towards the concept of superfield. We will consider the simplest case: an unconstrained, scalar superfield. We need first some mathematic terminology. In Appendix A, we recall the standard definitions of *sheaf* over a topological space and of *morphism of sheaves* (Definitions A.2 and A.5), which are used in the following. The reader interested in a more complete and deep treatment of the subject in the formalism that we use can consult for example Refs. [1, 2, 3, 4]).

Definition 3.1. A *superspace*³ $S = (|S|, \mathcal{O}_S)$ is a topological space $|S|$ endowed with a sheaf of superalgebras \mathcal{O}_S such that the stalk at each point $x \in |S|$, denoted as $\mathcal{O}_{S,x}$ is a local superalgebra (it has a unique maximal ideal). The sheaf \mathcal{O}_S is the *structural sheaf* of the superspace S . \square

The elements of $\mathcal{O}_S(U)$, for $U \subset_{\text{open}} |S|$ are *local sections over U* . If the open set is the total space $|S|$, then the elements of $\mathcal{O}_S(|S|)$ are called *global*

³The concept of superspace here is more general than the one used in physicist, that usually restricts to super spacetime.

sections. The superalgebra $\mathcal{O}_S(|S|)$ is also called the *coordinate superalgebra* of the superspace S .

Example 3.2. The *affine superspace* $k^{m|n}$ consists of the topological space k^m with the sheaf of superalgebras that for any open set $U \in k^m$ attaches the superalgebra $\mathcal{O}^{m|n}(U) := C^\infty(U) \otimes \wedge(\theta^1, \dots, \theta^n)$. □

Let x^1, \dots, x^m be global coordinates on k^m . Then we say that $x^1, \dots, x^m, \theta^1, \dots, \theta^n \in \mathcal{O}^{m|n}(k^m)$ are *global coordinates* on $k^{m|n}$. We recall that, as a general rule, Latin letters denote even (commuting) quantities and Greek letters denote odd (antim commuting) quantities.

It is important to pay attention to the definition of morphisms of superspaces. One can define them as morphisms of the corresponding sheaves but first one has to put them over the same basis. This is done by using the pull back sheaf.

Definition 3.3. A *morphism* $\phi : S \rightarrow T$ of superspaces is given by a pair $(|\phi|, \phi^\sharp :)$ where $|\phi| : |S| \rightarrow |T|$ is a homeomorphism and $\phi^\sharp : \mathcal{O}_T \rightarrow \phi_* \mathcal{O}_S$ is a morphism of sheaves that preserves the maximal ideal of the stalks. The sheaf $|\phi|_* \mathcal{O}_S$ is the pull back by $|\phi|$ of the sheaf \mathcal{O}_S , defined, for any open $U \subset T$, as

$$|\phi|_* \mathcal{O}_S(U) = \mathcal{O}_S(|\phi|^{-1}(U)).$$
□

Remark 3.4. It is not difficult to show that a morphism $\phi : k^{m|n} \rightarrow k^{p|q}$ is determined by the images of the global coordinates of $k^{p|q}$ under ϕ^\sharp (for a proof of this fact, see the *Chart Theorem*, Theorem 4.1.11, in Ref [3]). □

Definition 3.5. A *supermanifold* of dimension $m|n$ is a superspace that is locally isomorphic to $k^{m|n}$. □

Example 3.6. *Toy model.* We consider first the simplest model of superspace having both, even and odd components, the affine superspace $\mathbb{R}^{1|1} = (\mathbb{R}, \mathcal{O}^{1|1})$, with sheaf

$$\mathcal{O}_{\mathbb{R}}^{1|1}(U) = C^\infty(U)[\theta] = C^\infty(U) \otimes \wedge[\theta], \quad U \subset_{\text{open}} \mathbb{R}.$$

It is important to distinguish between the affine superspace $\mathbb{R}^{1|1}$ and the super vector space $V_{\mathbb{R}}^{1|1}$ of Example 2.5. On $V_{\mathbb{R}}^{1|1}$ we defined an algebra structure via the even rules principle which converts this space into the Grassmann algebra $\wedge(\theta)$. Tensoring with $C^\infty(U)$ one obtains $\mathcal{O}_{\mathbb{R}}^{1|1}(U)$.

We denote as x and θ the global coordinates on $\mathbb{R}^{1|1}$. Generically, an element of $\mathcal{O}^{1|1}(\mathbb{R})$ (a global section) can be written as

$$\tilde{\Phi} = \tilde{A} + \tilde{G}\theta, \quad (3)$$

with $\tilde{A}, \tilde{G} \in C^\infty(\mathbb{R})$. Notice that, formally, (3) looks like the superfields that appear in the physics literature, except for the fact that both, \tilde{A} and \tilde{G} , are ordinary (even) smooth functions on \mathbb{R} .

Let us denote $\mathcal{A}_{\mathcal{B}} = C^\infty(\mathbb{R}) \otimes (\mathcal{B} \otimes \wedge(\theta))_0$.

Definition 3.7. A *scalar superfield on the superspace $\mathbb{R}^{1|1}$* is a functorial family of elements

$$\{\Phi_{\mathcal{B}} = A + \chi\theta \in \mathcal{A}_{\mathcal{B}}, \quad A \in C^\infty(\mathbb{R}) \otimes \mathcal{B}_0, \chi \in C^\infty(\mathbb{R}) \otimes \mathcal{B}_1\}.$$

Since the family is functorial, it satisfies

$$\Phi_h(\Phi_{\mathcal{B}}) = \Phi_{\mathcal{B}'}.$$

□

A product, inherited for the products on $\mathcal{A}_{\mathcal{B}}$, is readily defined in the set of superfields.

To see how this works explicitly let us consider $\mathcal{B} = \wedge[\xi^1, \xi^2]$. Then

$$A(x) = A_0(x) + A_{12}(x)\xi^1\xi^2, \quad \chi(x) = \chi_1(x)\xi^1 + \chi_2(x)\xi^2.$$

We can now compute quantities as $(\dot{\chi} = d\chi/dx)$

$$\begin{aligned} \chi(x)\chi(x') &= (\chi_1(x)\chi_2(x') - \chi_2(x)\chi_1(x'))\xi^1\xi^2, \\ \chi(x)\dot{\chi}(x) &= (\chi_1(x)\dot{\chi}_2(x) - \chi_2(x)\dot{\chi}_1(x))\xi^1\xi^2, \end{aligned}$$

which, generically, are different from zero. These properties are used in classical field theory and have a (deformed, as $\hbar \neq 0$) counterpart in quantum field theory.

It is clear now that if one wants to have products of fields in n different points, or products of n fields and derivatives of fields that are not identically 0, one needs to increase the number of odd generators ξ^i up to n . In order to achieve full generality, one is then lead to consider all the Grassmann algebras, and from Remark 2.7, this is equivalent to the construction of the functorial family $\Phi = \{\Phi_{\mathcal{B}}, \mathcal{B} \in (\text{c salgebras})\}$.

□

Remark 3.8. Unlike the odd variable θ (the odd coordinate in the superspace $\mathbb{R}^{1|1}$), which opens the possibility of the odd component field χ , the Grassmann variables in $\wedge(\xi_1, \dots, \xi_n)$ are non physical, in the sense that they do not generate new fields but are used to reproduce the properties required for the already existing odd fields. The fact that there is a functorial behavior with respect to \mathcal{B} is a reflection of their spurious character. In Section 6 we will come back to this point.

□

One can easily extend the definition of scalar superfield in (3.7) for affine superspaces $k^{m|n}$.

Definition 3.9. A *scalar superfield in the affine space $k^{m|n}$* is a family of elements

$$\Phi = \{ \Phi_{\mathcal{B}} \in \mathbb{C}^\infty(k^m) \otimes (\mathcal{B} \otimes \wedge(\theta^1, \dots, \theta^n))_0, \quad \mathcal{B} \in (\text{c salgebras}) \} \quad (4)$$

which behaves functorially under a morphism $f : \mathcal{B} \rightarrow \mathcal{B}'$.

□

It should not offer special difficulty to take a local point of view and replace $\mathbb{C}^\infty(k^m)$ in (3.9) by $C^\infty(U)$, with $U \subset_{\text{open}} k^m$. In this way one could associate to each open subset U a family of algebras which behaves functorially in \mathcal{B} . These families are chosen so that they satisfy a gluing condition in the intersection of two open sets, since they arise from a sheaf of superalgebras. One could then extend the definition of scalar superfields to supermanifolds, which are modelled locally as affine superspaces.

It could be also useful to consider, instead of smooth functions, real analytic, complex holomorphic or polynomial functions.

Remark 3.10. Notice that we are not assuming, a priori, that there is an action of the super Poincaré group nor any other supergroup in superspace,

nor an invariance of the field theory under a supergroup. The concept of superfield, as we understand it here, is previous to any considerations about supersymmetry, and takes into account only the algebraic properties derived from the presence of odd coordinates. For example, a superfield could have only one component field, even or odd. Obviously, the subset of such superfields would never be enough to support the action of a supergroup with a non trivial odd part.

In the physics terminology, on the contrary, the word ‘superfield’ is inevitably linked to some supersymmetry transformations of the underlying superspace. Our definition is then more general. We can, at any moment, restrict to the subset of superfields that support a representation of some supergroup, and this set would be the adequate setting for describing supersymmetric field theories. But the algebraic properties that we want to reproduce here are indeed relevant from the very moment in which an odd field (say, an electron field) appears in the theory, irrespectively of its behaviour under supersymmetry transformations.

The adjective ‘scalar’ refers only to the fact that the superfields that we consider are associated with the structural sheaf (see Definition 3.1). Other types of superfields could be conveniently defined in terms of modules over the structural sheaf.

□

4 How to deal with algebraic constraints.

The examples that we would like to analyze are the result of some algebraic constraints imposed on a certain set of $N = 1$, $D = 4$ chiral superfields. We are then going to abandon the toy model for a slightly more complicated one. The superspace in this case is $\mathbb{C}^{4|2}$, the *chiral superspace*, with global sections $\mathcal{O}^{4|2}(\mathbb{C}^4) = C^\infty(\mathbb{C}^4)[\theta^1, \theta^2]$. A *chiral superfield* then is a functorial family $\Phi = \{\Phi_{\mathcal{B}}, \mathcal{B} \in (\text{c salgebras})\}$ of elements

$$\Phi_{\mathcal{B}} \in C^\infty(\mathbb{C}^4) \otimes (\mathcal{B} \otimes \wedge(\theta^1, \theta^2))_0. \quad (5)$$

An element there is written as

$$\Phi_{\mathcal{B}} = A_{\mathcal{B}} + \theta^\alpha \chi_{\mathcal{B}\alpha} + \theta^\alpha \theta_\alpha F_{\mathcal{B}}, \quad \alpha = 1, 2,$$

where the index notation is the usual in physics (and it is explained in Appendix B); in particular, sum over repeated indices is understood. Also, the

dependence on the even coordinates is not written explicitly. A and F are in $C^\infty(\mathbb{C}^4) \otimes \mathcal{B}_0$ and χ_α are in $C^\infty(\mathbb{C}^4) \otimes \mathcal{B}_1$. From now on we will omit the subscript \mathcal{B} and write

$$\Phi = A + \theta^\alpha \chi_\alpha + \theta^\alpha \theta_\alpha F, \quad \alpha = 1, 2,$$

We can still look at the superfield (3.9) in a slightly different way. We take the simpler approach where the commutative superalgebra \mathcal{B} runs over the superalgebras $\mathcal{B} = C^\infty(\mathbb{C}^4)[\xi^1, \dots, \xi^n]$, with arbitrary n (see Remark 2.7). We recall here also the Definition 3.1 of superspace morphisms $\phi : S \rightarrow T$. On the open set $U \subset |S|$, we have to give a superalgebra morphism

$$\phi^\# : \mathcal{O}_T(V) \rightarrow \mathcal{O}_S(U), \quad \phi(U) = V, .$$

For example, we can consider a morphism of superspaces

$$\mathbb{C}^{4|n} \xrightarrow{\phi} \mathbb{C}^{2|2}. \quad (6)$$

According to Remark 3.4, this is determined once we provide two even sections and two odd sections of $\mathbb{C}^{4|n}$, which are the images under $\phi^\#$ of the global coordinates in $\mathbb{C}^{2|2}$. Then, each superfield (5) provides, with its component fields (A, F, χ_1, χ_2) , a morphism as in (6).

We want now to study algebraic constraints on the set of chiral superfields (5) or, equivalently, on the morphisms (6). Spacetime dependence is untouched by the constraints that we consider, so one can effectively consider that spacetime is reduced to a point: one can recover the full picture by tensoring with $C^\infty(\mathbb{C}^4)$ where needed. What we will obtain are relations among the coordinates of the superspace $\mathbb{C}^{2|2}$. Since the constraints are not linear, the restricted space is not an affine superspace. Moreover, in some cases it can only be understood as a *superscheme*. In the next subsection we try to give a summary on the principal results on schemes and superschemes. In the non super case a complete treatment can be found in any textbook on algebraic geometry (see for example the first chapter of Ref. [6]). For the super case there is a thorough treatment in Ref. [3].

4.1 Schemes and superschemes

Affine algebraic varieties are commonly seen as the zero locus of some polynomials, although this description is not intrinsic, since it depends on a certain embedding. Modern algebraic geometry gives a different point of view.

An *affine algebra* F over an algebraically closed field (for example \mathbb{C}) is a commutative, finitely generated algebra that contains no nilpotents. In ordinary geometry, given an affine algebra one can construct a topological space, called the *spectrum of F* and denoted by $|X| = \text{Spec}(F)$, as the set of all prime ideals of F endowed with the *Zariski topology*.

One constructs over the topological space $|X|$ a sheaf of algebras by localizing F at each prime $\mathfrak{p} \in \text{Spec}(F)$. The algebra

$$F_{\mathfrak{p}} := \left\{ \frac{f}{g} \mid f \in F, g \in F - \mathfrak{p} \right\}$$

is the *stalk* of the sheaf at \mathfrak{p} and it is a *local algebra*, that is, it has a unique maximal ideal. The sheaf is denoted as \mathcal{O}_X or \mathcal{O}_F . Then, the pair $X = (|X|, \mathcal{O}_X)$ is an *affine algebraic variety*. The sheaf \mathcal{O}_X is the *structural sheaf* of the affine algebraic variety.

Making contact with the traditional point of view, $\text{Spec}(F)$ consists of the points of the algebraic variety (which correspond to the maximal ideals) together with all the irreducible subvarieties.

One recovers the affine algebra F as the set of global sections: $\mathcal{O}_X(X) = F$, and it is said to be the *coordinate ring* or *coordinate algebra* of the affine variety X .

It will be important to distinguish the pair $X = (|X|, \mathcal{O}_X)$ from the topological space $|X| = \text{Spec}(F)$, so we will denote

$$\underline{\text{Spec}}(F) := (\text{Spec}(F), \mathcal{O}_F) = (|X|, \mathcal{O}_X).$$

The procedure that we have described establishes an equivalence of categories between affine algebras and affine varieties. One can apply the same procedure to an algebra F that is not affine: $\text{Spec}(F)$ and \mathcal{O}_F still make sense. The nilpotent elements of a commutative algebra form an ideal N so one can define the *reduced algebra* $F_{\text{red}} := F/N$. Since N sits inside every prime ideal, we have that $\text{Spec}(F_{\text{red}}) \cong \text{Spec}(F)$ as topological spaces. Nevertheless, the sheaves \mathcal{O}_F and $\mathcal{O}_{F_{\text{red}}}$ will be, in general, different. In particular, F_{red} may be an affine algebra even if F is not.

Example 4.1. Let us consider the algebra of polynomials in two variables, $\mathbb{C}[x, y]$, and the ideal generated by the element x^2 . The quotient $F = \mathbb{C}[x, y]/\langle x^2 \rangle$ is not an affine algebra, since it contains a nilpotent, namely, the element x . A generic element of F will be of the form

$$f_0 + x f_1, \quad f_0, f_1 \in \mathbb{C}[y].$$

The solution of the polynomial equation $x^2 = 0$ over \mathbb{C} is just $x = 0$ and in fact, $F_{\text{red}} \cong \mathbb{C}[y]$. Nevertheless, the algebra F keeps track of the double multiplicity of the solution, so it has more information. The maximal ideals (that are also prime ideals) in $\text{Spec}(F)$ are of the form $\langle y - a, x^2 \rangle$, $a \in \mathbb{C}$ and the stalk of the sheaf at such point is

$$F_a = \left\{ \frac{f_0 + x f_1}{g_0 + x g_1} \mid f_i, g_i \in \mathbb{C}[y], \quad g_0(a) \neq 0 \right\}.$$

Working with the reduced algebra, the maximal ideals are $\langle y - a \rangle$, $a \in \mathbb{C}$, and the stalk is

$$F_{\text{red},a} = \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[y], \quad g(a) \neq 0 \right\}.$$

□

We are led to the following definition:

Definition 4.2. An *affine scheme*⁴ X is a topological space $|X|$ together with a sheaf of algebras \mathcal{O}_X which is isomorphic to $\underline{\text{Spec}}(F)$ for some algebra F .

□

Since superalgebras inevitably contain nilpotents, the concept of scheme seems suitable for extension to superalgebras. A superalgebra $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ is an *affine superalgebra* if its even part \mathcal{A}_0 is finitely generated as an algebra, its odd part \mathcal{A}_1 is finitely generated as an \mathcal{A}_0 -module and the *reduced algebra* defined as $\mathcal{A}_{\text{red}} = \mathcal{A}/\mathcal{J}$, where \mathcal{J} is the ideal of odd nilpotents, is itself affine, so it contains no further nilpotents. Notice that we quotient here only by the odd nilpotents, that is, the nilpotents that are present in any superalgebra. There may remain some even nilpotents as in Example 4.1. Nevertheless it is customary to call to \mathcal{A}_{red} the *reduced algebra of the superalgebra* \mathcal{A} . So one can have superalgebras whose reduced algebra is not affine: they are then non affine superalgebras.

One can also construct the topological space $\text{Spec}(\mathcal{A}) = \text{Spec}(\mathcal{A}_{\text{red}})$, and equip it with a sheaf of superalgebras obtained by localization. We then have:

⁴It may seem odd that the category of affine schemes relates to non affine algebras, but the adjective ‘affine’ on the noun ‘scheme’ is used to distinguish it from a *projective scheme*, a generalization that we will not need in this paper.

Definition 4.3. An *affine superscheme* is a superspace $S = (|S|, \mathcal{O}_S)$ which is isomorphic to $\underline{\text{Spec}}(\mathcal{A})$ for a superalgebra \mathcal{A} not necessarily affine. \square

Given an affine superscheme S with superalgebra \mathcal{A} there is always an affine scheme S_{red} associated to the reduced algebra \mathcal{A}_{red} . It is the *reduced scheme* of the superscheme, a concept which is similar to the concept of reduced manifold of a supermanifold or reduced algebraic variety of an algebraic supervariety.

4.2 Constraint $\Phi^2 = 0$

This first example of constrained superfield bears some resemblance with Example 4.1.

If one, naively, considers the constraint $\tilde{\Phi}^2 = 0$ on sections of the structural sheaf of $\mathbb{C}^{2|2}$ (see (3))

$$\tilde{\Phi} = \tilde{A} + \theta^\alpha \tilde{G}_\alpha + \theta^\alpha \theta_\alpha \tilde{F}, \quad \alpha = 1, 2,$$

where \tilde{A} , \tilde{G}_α and \tilde{F} are all (even) functions in $C^\infty(\mathbb{C}^4)$, then

$$\tilde{\Phi}^2 = \tilde{A}^2 + 2\tilde{A}\theta^\alpha \tilde{G}_\alpha + 2\tilde{A}\tilde{F}\theta^\beta \theta_\beta = 0.$$

If we assume that F is invertible, then $\tilde{A} = 0$ and \tilde{G}_α is free.

If, instead, we consider the constraint $\Phi^2 = 0$ on the superfield

$$\Phi^2 = A^2 + 2A\theta^\alpha \chi_\alpha + (2AF - \frac{1}{2}\chi^\alpha \chi_\alpha)\theta^\beta \theta_\beta = 0,$$

where $A, F \in \mathcal{B}_0$ and $\chi_\alpha \in \mathcal{B}_1$, this gives the system of equations

$$A^2 = 0, \quad A\chi_\alpha = 0, \quad 4AF - \chi^\alpha \chi_\alpha = 0, \quad (7)$$

which define an affine superscheme denoted as S . The superalgebra of global sections, $\mathcal{O}(S)$ (shorter notation for $\mathcal{O}_S(|S|)$) is

$$\mathcal{O}(S) = C^\infty(\mathbb{C}^2)[\chi_1, \chi_2] / \langle A^2, A\chi_\alpha, 2AF - \chi_1\chi_2 \rangle, \quad (A, F) \in \mathbb{C}^2.$$

(Notice that last equation in (7) can be also written as $2AF - \chi_1\chi_2 = 0$).

The reduced (affine) scheme S_{red} is obtained by setting the odd variables to zero, which gives quadratic relations

$$A^2 = 0, \quad AF = 0, \quad (8)$$

so the algebra defining the affine scheme is

$$\mathcal{O}(S_{\text{red}}) = C^\infty(\mathbb{C}^2) / \langle A^2, AF \rangle, \quad (A, F) \in \mathbb{C}^2.$$

Since A is an even nilpotent, the scheme is not an algebraic variety.

There is, however, an open set where the scheme is isomorphic to an affine one. This corresponds to the points where F is invertible, that is, to the prime ideals of $\mathcal{O}(S_{\text{red}})$ that do not contain F . We will denote the localization of $\mathcal{O}(S_{\text{red}})$ at these points as $\mathcal{O}(S_{\text{red}})_{F \neq 0}$. This essentially means that we can set $A = 0$ from the second equation in (8); then, the first one is satisfied identically:

$$\mathcal{O}(S_{\text{red}})_{F \neq 0} \simeq C^\infty(\mathbb{C}^\times). \quad (9)$$

This is the regular or smooth part of the scheme, represented by the object $\mathbb{C}^\times = \mathbb{C} - \{0\}$.

Going back to the superscheme, we can do a change of variables for the coordinates:

$$A' = 4AF - \chi^\alpha \chi_\alpha, \quad F' = F \quad \chi'_\alpha = \chi_\alpha, \quad (10)$$

with Jacobian

$$J = \begin{pmatrix} 4F & 4A & -2\chi_2 & +2\chi_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is non singular for $F \neq 0$. The inverse transformation is

$$A = \frac{A' + \chi^\alpha \chi_\alpha}{4F}, \quad F = F', \quad \chi_\alpha = \chi'_\alpha,$$

and the smooth part of the superscheme is just given by the ring

$$C^\infty(\mathbb{C}^2)[\chi_1, \chi_2]_{F \neq 0} / \langle A' \rangle \simeq C^\infty(\mathbb{C}^\times)[\chi_1, \chi_2], \quad (A', F) \in \mathbb{C}^2.$$

This model appeared in its non linear form in Ref. [7], so it is called the *Volkov-Akulov multiplet*. In terms of superfields appeared in Refs. [8, 9, 10].

4.3 General constraint $f(\Phi) = 0$

Let f be a polynomial in one variable. We consider now the more general constraint

$$f(\Phi) = 0.$$

Because of the nilpotency of θ^α , this reduces to

$$f(\Phi) = f(A) + \theta^\alpha \chi_\alpha f'(A) + \theta^\alpha \theta_\alpha \left(f'(A)F - \frac{1}{4} f''(A) \chi^\alpha \chi_\alpha \right).$$

For example, let us take $f(\Phi) = \Phi^n$. Then the constraints that define the affine superscheme S are:

$$A^n = 0, \quad A^{n-1} \chi_\alpha = 0, \quad A^{n-2} (4AF - (n-1) \chi^\alpha \chi_\alpha) = 0. \quad (11)$$

The reduced affine scheme in this case is defined by the ring

$$\mathcal{O}(S_{\text{red}}) = C^\infty(\mathbb{C}^2) / \langle A^n, A^{n-1}F \rangle, \quad (A, F) \in \mathbb{C}^2.$$

We can still localize at F invertible and we get

$$\mathcal{O}(S_{\text{red}})_{F \neq 0} = C^\infty(\mathbb{C}^2)_{F \neq 0} / \langle A^{n-1} \rangle \simeq C^\infty(\mathbb{C} \times \mathbb{C}^\times) / \langle A^{n-1} \rangle.$$

Differently to (9) this ring still has the nilpotent A everywhere, so it does not have smooth points.

Remark 4.4. Following what some authors in physics do, one could hand-impose an extra constraint

$$A = a \chi^\alpha \chi_\alpha. \quad (12)$$

Here a is a coefficient that can depend on F but not on χ^α , because then A would be identically zero.

For $n = 2$ this followed from the constraints (7) only assuming that F was invertible. Then a was determined to be $a = 1/4F$. The same trick would not work for $n \geq 3$, so (12) is an extra constraint, not coming from (11), which nevertheless allows to solve trivially (11). Moreover, a is arbitrary, so there are indeed solutions to $\Phi^3 = 0$ that do not solve $\Phi^2 = 0$. For $n = 3, 4, 5, \dots$ the sets of solutions that we obtain in this way are identical.

The constraint (12), when projecting onto the reduced algebra (putting the fermions to zero), gives $A = 0$, which leaves us with the affine algebra $C^\infty(\mathbb{C})$, something similar to what happened in Example 4.1. The superalgebra would be $C^\infty(\mathbb{C})[\chi_1, \chi_2]$, that is, the algebra of the affine superspace $\mathbb{C}^{1|2}$.

□

This type of constraints appear in Ref. [11].

Remark 4.5. In Appendix C we wrote the infinitesimal supertranslation algebra which acts on the affine superspace $\mathbb{C}^{4|2}$. Actually, the supertranslation generators (26) act on $\mathbb{C}^{4|4}$, the *complexified Minkowski superspace*. There is of course a real version of this superspace and of the supertranslation algebra which is the usual in physics. On the *chiral superspace* $\mathbb{C}^{4|2}$ only acts the superalgebra generated by the generators P_μ and Q_α .

The sets of equations (7) and (11) are supersymmetric since $\Phi^2 = 0$ is a supersymmetric constraint. While in the $n = 2$ case the solution obtained by inverting F is a supersymmetric solution, in the $n \geq 3$ case the solution obtained by imposing (12) is not supersymmetric. This can be checked by explicit calculation. Nevertheless, being the constraints supersymmetric, the space of solutions of (11) has an action of the supertranslation algebra. It is then mandatory to keep the nilpotent A with $A^n = 0$ in order to preserve supersymmetry. Although we do not know yet the physical interpretation of such fields, it is remarkable that one is lead to maintain genuinely even nilpotents (that is, nilpotents that survive the projection onto the reduced scheme) in order to preserve the supersymmetry. □

In this situation we do not have enough odd variables for the problem with $n > 3$ to be interesting. One can add more odd variables by going to extended supersymmetry. Physically, though, it is more difficult to give meaning to superspace and superfields in extended supersymmetry. We could also consider real superfields, which have four real odd variables. Finally, what has been done in the literature is to use several superfields.

In the next section we see how we can satisfy cubic constraints with two superfields.

4.4 Cubic constraint with two superfields

Let us start with two superfields

$$\Phi_1 = A_1 + \theta^\alpha \chi_{1\alpha} + \theta^\alpha \theta_\alpha F_1, \quad \Phi_2 = A_2 + \theta^\alpha \chi_{2\alpha} + \theta^\alpha \theta_\alpha F_2.$$

The quantities A_1, A_2, F_1, F_2 (even) and $\chi_{1\alpha}, \chi_{2\alpha}$, $\alpha = 1, 2$ (odd) are coordinates in the superspace $\mathbb{C}^{4|4}$. On these coordinates we want to impose the constraint $\Phi_1 \cdot \Phi_2 = 0$:

$$\Phi_1 \cdot \Phi_2 = A_1 A_2 + \theta^\alpha (A_1 \chi_{2\alpha} + A_2 \chi_1^\alpha) + \theta^\alpha \theta_\alpha (A_1 F_2 + A_2 F_1 - \frac{1}{2} \chi_1^\alpha \chi_{2\alpha}) = 0,$$

which implies

$$\begin{aligned} A_1 A_2 &= 0, \\ A_1 \chi_{2\alpha} + A_2 \chi_{1\alpha} &= 0, \\ A_1 F_2 + A_2 F_1 - \frac{1}{2}(\chi_1 \chi_2) &= 0. \end{aligned} \tag{13}$$

To ease the notation we write $(\chi_1 \chi_2) := \chi_1^\alpha \chi_{2\alpha}$. This defines an affine superscheme S with superalgebra

$$\mathcal{O}(S) := C^\infty(\mathbb{C}^4)[\chi_{1\alpha}, \chi_{2\alpha}] / \langle A_1 A_2, A_1 \chi_{2\alpha} + A_2 \chi_{1\alpha}, A_1 F_2 + A_2 F_1 - \frac{1}{2}(\chi_1 \chi_2) \rangle.$$

Let us compute the reduced part of the scheme, S_{red} . Setting to zero the odd coordinates in (13) we get

$$A_1 A_2 = 0, \quad A_1 F_2 + A_2 F_1 = 0, \tag{14}$$

so

$$\mathcal{O}(S_{\text{red}}) = C^\infty(\mathbb{C}^4) / \langle A_1 A_2, A_1 F_2 + A_2 F_1 \rangle.$$

We can now localize at $F_1 \neq 0$ and solve for A_2 . Then (14) becomes

$$F_2 A_1^2 = 0, \quad A_2 = -F_1^{-1} F_2 A_1.$$

Restricting also to the points $F_2 \neq 0$ we get

$$A_1^2 = 0, \quad A_2 = -F_1^{-1} F_2 A_1,$$

so A_1 and A_2 are even nilpotents. The ring then becomes

$$\mathcal{O}(S_{\text{red}})_{F_1, F_2 \neq 0} = C^\infty(\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}) / \langle A_1^2 \rangle, \quad A_1 \in \mathbb{C},$$

and the scheme is not regular.

We now reintroduce the odd variables. Localizing at $F_1 \neq 0$ we can solve for A_2

$$A_2 = -\frac{F_2}{F_1} A_1 + \frac{1}{2F_1}(\chi_1 \chi_2).$$

Inserting into the first and second equations in (13) we get

$$\begin{aligned} F_2 A_1^2 - \frac{1}{2}(\chi_1 \chi_2) A_1 &= 0 \\ A_1 \left(\chi_{2\alpha} - \frac{F_2}{F_1} \chi_{1\alpha} + \frac{1}{2F_1}(\chi_1 \chi_2) \chi_{1\alpha} \right) &= 0. \end{aligned} \tag{15}$$

If $F_2 \neq 0$, one can see that $A_1^4 = 0$, so A_1 is nilpotent.

One may consider, as in (12), the extra condition that A_1 is an even function of the odd variables,

$$A_1 = a(\chi_1^2) + b(\chi_2^2) + c(\chi_1\chi_2), \quad (16)$$

with a , b and c coefficients that can also be functions of the other fields. As before, we stress that the meaning is that when projecting onto the reduced algebra we take $A_1 = 0$.

Inserting now the ansatz in (15), and after some calculations, we get

$$a = \frac{1}{4F_1} - \frac{F_2^2}{F_1^2} b, \quad c = -\frac{2F_2}{F_1} b,$$

and b is free. We have made use of the identities

$$(\chi_1\chi_2)\chi_{1\alpha} = -\frac{1}{2}(\chi_1^2)\chi_{2\alpha}, \quad (\chi_1\chi_2)\chi_{2\alpha} = -\frac{1}{2}(\chi_2^2)\chi_{1\alpha}.$$

So the superfields become

$$\begin{aligned} A_1 &= \left(\frac{1}{4F_1} + \frac{F_2^2}{F_1^2} b \right) (\chi_1^2) + b(\chi_2^2) - 2\frac{F_2}{F_1} b(\chi_1\chi_2), \\ A_2 &= \left(-\frac{F_2}{4F_1^2} - \frac{F_2^3}{F_1^3} b \right) (\chi_1^2) - \left(2\frac{F_2^2}{F_1^2} b - \frac{F_2}{F_1} b(\chi_2^2) + \frac{1}{2F_1} \right) (\chi_1\chi_2), \end{aligned} \quad (17)$$

with χ_1 and χ_2 free, F_2 free, and $F_1 \neq 0$. Since the equations (13) are symmetric under the exchange $1 \rightleftharpoons 2$, one can obtain a similar solution by inverting F_2 .

If $b = 0$, the terms proportional to b in Φ_1 vanish and we get $\Phi_1^2 = 0$. With this choice, also for Φ_2 the terms proportional to b disappear but

$$\Phi_2^2 = -\frac{1}{8F_1} \chi_1^2 \chi_2^2 \neq 0.$$

It is not difficult to see that the particular set of solutions with $b = 0$ are equivalent to the system

$$\Phi_1^2 = 0, \quad \Phi_1\Phi_2 = 0, \quad (18)$$

once we have inverted F_1 . One obtains

$$A_1 = \frac{1}{4F_1}(\chi_1)^2, \quad A_2 = \frac{(\chi_1\chi_2)}{2F_1} - \frac{F_2(\chi_1)^2}{4F_1^2},$$

and the remaining coordinates free (note that F_2 is not required to be invertible). This system has supersymmetry.

We check now the cubic constraints for generic b :

$$\Phi_1^2\Phi_2 = \Phi_1\Phi_2^2 = 0$$

by virtue of $\Phi_1\Phi_2 = 0$. On the other hand, one gets also

$$\Phi_1^3 = 0, \quad \Phi_2^3 = 0.$$

From the three equations in (11) with $n = 3$, the first two ones are trivially satisfied because they are order greater than four in the fermionic variables and we only have four of them. The third one is only of order four in the fermionic variables and the terms must cancel exactly. It is not difficult to check that this happens for both superfields.

Nevertheless, the system

$$\Phi_1\Phi_2 = 0, \quad \Phi_1^3 = 0, \quad \Phi_2^3 = 0$$

gives rise to a superscheme that is not regular. The constraints are

$$\begin{aligned} A_1A_2 &= 0, & A_1\psi_2 + A_2\psi_1 &= 0, & A_1F_2 + A_2F_1 - \frac{1}{2}(\psi_1\psi_2) &= 0, \\ A_1^3 &= 0, & A_2^3 &= 0, & A_1^2\psi_{1\alpha} &= 0, & A_2^2\psi_{2\alpha} &= 0, \\ A_1 \left(A_1F_1 - \frac{1}{2}(\psi_1)^2 \right) &= 0, & A_2 \left(A_2F_2 - \frac{1}{2}(\psi_2)^2 \right) &= 0. \end{aligned}$$

The reduced scheme is given by the constraints

$$A_1A_2 = 0, \quad A_1F_2 + A_2F_1 = 0, \quad A_1^3 = 0, \quad A_2^3 = 0, \quad F_1A_1^2 = 0, \quad F_2A_2^2 = 0.$$

Even restricting to $F_1 \neq 0$, one obtains

$$A_2 = -\frac{F_2}{F_1}A_1, \quad A_1^2 = 0,$$

so a nilpotent remains that cannot be put directly to zero. The solutions obtained in (17) by imposing the ansatz (16) do not reflect the whole solution space, and consequently they are not supersymmetric.

These constraints appeared in Refs. [12, 10, 13].

4.5 Cubic constraint with an arbitrary number of superfields

The system (18) can be generalized by adding more chiral superfields. We consider the following system :

$$\begin{aligned} X &= A + (\theta\chi) + \theta^2 F, & X^2 &= 0 \\ Y_i &= A_i + (\theta\psi_i) + \theta^2 F_i, & XY_i &= 0 \end{aligned}$$

for $i = 1, \dots, n$ and n arbitrary. From (7) and (13) the constraints are equivalent to the system

$$\begin{aligned} A^2 &= 0, & A\chi_\alpha &= 0, & 4AF - \psi^2 &= 0 \\ AA_i &= 0, & A\psi_{i\alpha} + A_i\chi_\alpha &= 0, & AF_i + A_iF - \frac{1}{2}(\chi\psi_i) &= 0. \end{aligned}$$

Putting the fermions to zero, the constraints on the reduced part become

$$\begin{aligned} A^2 &= 0, & AF &= 0 \\ AA_i &= 0, & AF_i + A_iF &= 0. \end{aligned}$$

Localizing at F invertible we can solve

$$A = 0, \quad A_i = 0,$$

which means that the reduced scheme has a smooth part

$$\mathbb{C}[S_{\text{red}}]_{F \neq 0} = \mathbb{C}^\infty(\mathbb{C}^\times \times \mathbb{C}^n).$$

Reinserting the fermions we get

$$A = \frac{\psi^2}{4F}, \quad A_i = -\frac{\chi^2 F_i}{4F^2} + \frac{1}{2F}(\chi\psi_i),$$

and the remaining equations are satisfied trivially.

In this case we can use the same method than in Section 4.2. We perform a change of variables

$$\begin{aligned} A' &= 4AF - \chi^2, & A'_i &= AF_i + A - iF - \frac{1}{2}\chi\psi_i, \\ F' &= F, & F'_i &= F_i, & \chi' &= \chi, & \psi'_i &= \psi_i, \end{aligned}$$

whose Jacobian is invertible if F is so. The constraints are then

$$A' = 0, \quad A'_i = 0,$$

and the superscheme at F invertible becomes

$$\mathbb{C}[\mathcal{N}]_{F \neq 0} \simeq \mathbb{C}^\infty(\mathbb{C}^\times \times \mathbb{C}^n)[\chi_\alpha, \psi_{i\alpha}],$$

with $\alpha = 1, 2$, and $i = 1, \dots, n$. The superfields Y_i , $i = 1, \dots, n$ satisfy

$$Y_i Y_j Y_k = 0, \quad \forall i, j, k = 1, \dots, n. \quad (19)$$

In order to prove this we have used the following Fierz identity:

$$(\psi_1 \psi_2) \psi_3 + (\psi_3 \psi_1) \psi_2 + (\psi_2 \psi_3) \psi_1 = 0.$$

We note that this is not the most general solution to (19). For $n = 1$ the solution presented in (17) with $\Phi_1 = X$ and $\Phi_2 = Y$ is a more general one ($b \neq 0$).

The case $n = 3$ was presented in Ref. [14, 15].

5 A non algebraic constraint

In this section we are going to consider both, chiral and antichiral superfields. Up to now we were considering only chiral superfields, so the description of Section 4 was the simplest one. Moreover, the spacetime coordinates would not appear explicitly in the discussion, so it was as if spacetime was reduced to a point. In this section we will not be in that case anymore and the spacetime variables would play a role.

We shall start with a (complexified) super spacetime $\mathbb{C}^{4|4}$, with

$$x^\mu, \quad \mu = 0, \dots, 3, \quad \theta^\alpha, \bar{\theta}^{\dot{\alpha}}, \quad \alpha, \dot{\alpha} = 1, 2$$

being its global coordinates. As the notations suggests, θ^α and $\bar{\theta}^{\dot{\alpha}}$ are related by an antilinear involution that defines the standard (real) super Minkowski space and for which the coordinates x^μ are real.

In Refs. [16, 4], chiral and antichiral spaces are seen to be related to certain supergrassmannians, while the superspace having the correct real form is a superflag manifold. The superspaces $\mathbb{C}^{4|2}$ and $\mathbb{C}^{4|4}$ are only the bigcells of the above mentioned supermanifolds.

But we do not need to discuss here this interpretation, so we stick to work in the big cells which are affine superspaces.

On the superalgebra global sections of $\mathbb{C}^{4|4}$, $\mathcal{O}^{4|4}(\mathbb{C}^4) = C^\infty(\mathbb{C}^4) \otimes \wedge^4$, we can define two derivations

$$D_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu,$$

where

$$\partial_\alpha = \frac{\partial}{\partial \theta^\alpha}, \quad \partial_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}, \quad \partial_\mu = \frac{\partial}{\partial x^\mu},$$

and the Pauli matrices are given in Appendix B. D_α and $\bar{D}_{\dot{\alpha}}$ are indeed the right invariant vector fields of the action of the supertranslation generators (see Appendix C).

In physics, one defines a chiral superfield as a section of $\mathbb{C}^{4|4}$ such that

$$\bar{D}_{\dot{\alpha}} X = 0.$$

It is an easy calculation to see that, under the change of variables

$$(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \longrightarrow (y^\mu = x^\mu + i(\theta\sigma^\mu\bar{\theta}), \theta^\alpha, \bar{\theta}^{\dot{\alpha}}), \quad (\theta\sigma^\mu\bar{\theta}) := \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}},$$

a chiral superfield can be written as

$$X^{\text{ch}} = A^{\text{ch}}(y) + (\theta\chi^{\text{ch}}(y)) + \theta^2 F^{\text{ch}}(y).$$

Instead, and under the change of variables

$$(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \longrightarrow (\bar{y}^\mu = x^\mu - i(\theta\sigma^\mu\bar{\theta}), \theta^\alpha, \bar{\theta}^{\dot{\alpha}}),$$

the antichiral superfield

$$D_\alpha X^{\text{ach}} = 0,$$

is expressed as

$$X^{\text{ach}} = A^{\text{ach}}(\bar{y}) + (\bar{\theta}\chi^{\text{ach}}(\bar{y})) + \bar{\theta}^2 F^{\text{ach}}(\bar{y}).$$

y^μ and \bar{y}^μ are related by complex conjugation (as the notation suggests). Also, the complex conjugate of a chiral superfield is an antichiral superfield. We also have

$$\bar{y}^\mu = y^\mu - 2i(\theta\sigma^\mu\bar{\theta}). \quad (20)$$

Let us now consider now two chiral superfields

$$X = A(y) + \theta\chi(y) + \theta^2 F(y), \quad Y = B(y) + \theta\psi(y) + \theta^2 G(y),$$

and assume that $X^2 = 0$ as in Section 4.2. Then, if F is invertible

$$X = \frac{\chi(y)^2}{4F(y)} + \theta\chi(y) + \theta^2 F(y).$$

We write the complex conjugate of X as

$$\bar{X} = \frac{\bar{\chi}(\bar{y})^2}{4\bar{F}(\bar{y})} + \bar{\theta}\bar{\chi}(\bar{y}) + \theta^2 \bar{F}(\bar{y}),$$

where the notation (the usual one in physics) means

$$\bar{A}(\bar{y}) := \overline{A(y)}, \quad \bar{\chi}_{\dot{\alpha}}(\bar{y}) := \overline{\chi_{\alpha}(y)}, \quad \bar{F}(\bar{y}) := \overline{F(y)}.$$

The constraint that we intend to impose is [10]

$$\bar{X}Y = \text{antichiral}. \quad (21)$$

In order to do that, one writes the superfield Y in terms of the variable \bar{y}^{μ} , using (20) and expanding in Taylor series, which is finite because of the nilpotency of the odd variables. One gets

$$Y = B + \theta\psi + \theta^2 G + 2i\partial_{\mu}B(\theta\sigma^{\mu}\bar{\theta}) - i\theta^2(\partial_{\mu}\psi\sigma^{\mu}\bar{\theta}),$$

where all the component fields are evaluated at \bar{y}^{μ} . In order to impose the constraint (21), the only components of $\bar{X}Y$ that can survive are the ones proportional to 1, $\bar{\theta}^{\alpha}$ and $\bar{\theta}^2$. After some calculations we get (recall that F is invertible)

$$\begin{aligned} \bar{\chi}^2\psi_{\alpha} &= 0, & \bar{\chi}^2 G &= 0 \\ \frac{\partial^2 B}{4\bar{F}}\bar{\chi}^2 + \frac{i}{2}(\partial_{\mu}\psi\sigma^{\mu}\bar{\chi}) + \bar{F}G &= 0, & -i\partial_{\mu}B\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} + \psi_{\alpha}\bar{F} &= 0, \\ -i(\partial_{\mu}\psi^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}})\frac{\bar{\chi}^2}{4\bar{F}} + \bar{\chi}_{\dot{\alpha}}G &= 0, & \frac{i\partial_{\mu}B\chi^2}{2\bar{F}}\sigma^{\mu}_{\alpha\dot{\alpha}} + \bar{\chi}_{\dot{\alpha}}\psi_{\alpha} &= 0. \end{aligned} \quad (22)$$

Putting the fermions to zero, we get $G = 0$ and B undetermined. The constraints are algebraic and the reduced scheme is

$$\mathcal{O}(S_{\text{red}}) \simeq C^\infty(\mathbb{C}^\times \times \mathbb{C}).$$

The full constraints can be considerably simplified using that F is invertible. For example, one can isolate G and ψ_α

$$G = -\frac{\partial^2 B}{4\bar{F}^2} \bar{\chi}^2 - \frac{i}{2\bar{F}} (\partial_\mu \chi \sigma^\mu \bar{\chi}), \quad \psi_\alpha = i \partial_\mu B \sigma_{\alpha\dot{\alpha}}^\mu \bar{\chi}^{\dot{\alpha}},$$

and the remaining constraints are satisfied. Although the constraints involve derivatives, the superscheme can be given algebraically, in its complex version as

$$\mathcal{O}(S) \simeq C^\infty(\mathbb{C}^\times \times \mathbb{C})[\chi_\alpha, \bar{\chi}_{\dot{\alpha}}].$$

This example is also illustrative of the properties of the fermionic fields. All through the calculations one has to assume that a fermionic field, say χ_α , and its spacetime derivative $\partial_\mu \chi_\alpha$ have a product that is different from zero. Finally, the superscheme has a smooth part, on which the supersymmetry transformations have a well defined action.

6 Observables and nilpotent variables

Let F be an algebra and consider the scheme $\text{Spec}(F)$. Let \mathfrak{p} be a prime ideal in F , so $\mathfrak{p} \in |X| = \text{Spec}(F)$ and consider the quotient F/\mathfrak{p} . This is an integral domain (the product of two non zero elements is a non zero element). Moreover, if we consider the localization of F over \mathfrak{p} , $F_{\mathfrak{p}}$, and we quotient with the ideal $F_{\mathfrak{p}} \cdot \mathfrak{p}$, the result $\kappa(\mathfrak{p}) := F_{\mathfrak{p}}/(F_{\mathfrak{p}} \cdot \mathfrak{p})$ is a field, since every non zero element in $\kappa(\mathfrak{p})$ has an inverse. The field $\kappa(\mathfrak{p})$ is called the *residue field of $|X|$ at \mathfrak{p}* .

Example 6.1.

1. We consider the ring of polynomials in one variable $F = \mathbb{C}[x]$. The prime ideals of F are of the form $\mathfrak{p}_a = \langle x - a \rangle$, $a \in \mathbb{C}$ or the ideal $\langle 0 \rangle$. It is not difficult to see that the residue field at \mathfrak{p}_a is $\kappa(\mathfrak{p}_a) \cong \mathbb{C}$ and $\kappa(\langle 0 \rangle)$ is the field of rational functions.

2. If the ground field is \mathbb{R} , we have that every irreducible polynomial in $\mathbb{R}[x]$ generates a prime ideal. We still have the maximal ideals as $\mathfrak{p}_a = \langle x - a \rangle$ that give all the points of \mathbb{R} . At them, the residue field is $\kappa(\mathfrak{p}_a) \cong \mathbb{R}$. But, for example, the irreducible polynomial $x^2 + 1$ also generates a prime ideal. It is not difficult to realize that the elements of the residue field are of the form $a + xb$, with $a, b \in \mathbb{R}$ and x such that $x^2 + 1 = 0$, so $\kappa(\langle x^2 + 1 \rangle) \cong \mathbb{C}$.

□

As we have seen, affine schemes have residue fields that can vary from point to point. Let F be an algebra, not necessarily affine. For every element $f \in F$ we can define a ‘function’ on $\text{Spec}(F)$ with values in the residue field via the canonical maps

$$\begin{aligned} F &\longrightarrow F_{\mathfrak{p}} \longrightarrow \kappa(\mathfrak{p}) \\ f &\longrightarrow f \longrightarrow f(\mathfrak{p}). \end{aligned}$$

In an affine variety, with affine algebra F , one recovers in this way the original interpretation of F as the algebra of functions on the algebraic variety. The same holds in the case of differentiable manifolds and smooth functions.

If F contains a nilpotent element, say n , then $n \in \mathfrak{p}$ for all prime ideals \mathfrak{p} , so $n(\mathfrak{p}) = 0$. In other words, n is sent to the zero function and one cannot reproduce the original algebra F starting from an algebra of functions on $\text{Spec}(F)$. This is something that we already knew (see Example 4.1), but now we can read it from a physical point of view.

A classical mechanics system is commonly described in terms of an symplectic manifold called *phase space*, whose points represent the possible states of the system. Classical observables are smooth functions on phase space. There is a special observable, the Hamiltonian, which governs the time evolution of the system: given the initial state in an instant of time t_0 , the system evolves in future times t by following the integral curve of the hamiltonian vector field associated to the Hamiltonian, passing through the initial state.

This picture can be more or less carried over classical field theory by substituting the phase space for an infinite dimensional space of maps from spacetime to a target manifold or of sections of some bundle over spacetime, which are the *fields*. Most of the time one uses variational calculus to approach classical field theory instead of trying to give some comprehensive study of these infinite dimensional spaces, which can be very involved.

Nevertheless, the idea of observable is mimicked from classical mechanics: observables are (a special class of) functionals on the space of fields. The time evolution of the system is also governed by some partial differential equations (usually second order) for the fields.

The idea that we want to convey can be already understood at the classical mechanics level. Suppose that we want to generalize the classical phase space to some sort of affine scheme whose algebra contains nilpotents. The usual way to obtain ‘numbers’ (results of a measurement) from the sections of the scheme is by the evaluation procedure explained above. Nilpotent elements go to zero by this map, so they do not represent observables.

Now, one could do the same for a superscheme: even a smooth supermanifold or regular algebraic variety contains nilpotents generated by the odd elements which can not be seen in any measurement. What we are affirming is that, at least in this interpretation of observable, classical, odd degrees of freedom could not be seen in experiments.

In quantum mechanics things are very different. States are rays in a Hilbert space and observables are hermitian operators on it. The results of measurements are eigenvalues of these operators, and they appear with a probability distribution determined by the Hilbert space state. The algebra of operators on a Hilbert space is non commutative, so sometimes it is said, very roughly, that ‘quantizing’ a system corresponds to substitute the commutative algebra of observables by a non commutative one such that when taking the limit $\hbar \rightarrow 0$ the original commutative algebra is recovered.

Let us consider the simplest case possible, a two dimensional phase space \mathbb{R}^2 with canonical coordinates $(q, p) \in \mathbb{R}^2$ and symplectic form $dq \wedge dp$. The induced Poisson bracket on the coordinates is

$$\{q, p\}_- = 1. \quad (23)$$

As a quantum system, one considers the Hilbert space of square integrable functions on the variable q , $\mathcal{L}^2(\mathbb{R})$. One considers the position and momentum operators:

$$Qf(q) = qf(q), \quad Pf(q) = -i\hbar \frac{\partial f}{\partial q},$$

whose commutation rule is

$$[Q, P]_- = i\hbar \text{id}.$$

Taking $\hbar \rightarrow 0$ the commutation relation is reverted to the commutativity of q and p as ordinary functions on phase space. The fact that the term of order one in \hbar is proportional to the Poisson bracket (changing the constant function ‘1’ by the identity) is not casual, but a requirement of the quantization.

Let us assume now that the phase space is substituted by a superspace, for example $\mathbb{R}^{2|2}$, with superalgebra $C^\infty(\mathbb{C}^2) \otimes \wedge(\theta, \pi)$. There is also a super Poisson structure on it. The Poisson bracket of two odd quantities is symmetric

$$\{\theta, \pi\}_+ = 1, \quad \{q, p\}_- = 1,$$

and the rest zero.

As in the non super case, the superalgebra $C^\infty(\mathbb{C}^2) \otimes \wedge(\theta, \pi)$ admits a deformation with parameter \hbar . We focuss exclusively on the deformation of the Grassmann algebra $\wedge(\theta, \pi)$. Mimicking the procedure with the even variables, we get a non commutative superalgebra with generators Θ and Π satisfying the commutation rules

$$[\Theta, \Pi]_+ = i\hbar \text{id} \quad [\Theta, \Theta]_+ = 0, \quad [\Pi, \Pi]_+ = 0,$$

which is the algebra of the fermionic quantum oscillator with its creation and destruction operators. From them, one can construct the Hilbert space and the quantum observables.

There is a linear change of variables

$$\Gamma = \Theta + i\Pi, \quad \Upsilon = \Theta - i\Pi,$$

which shows that this algebra is isomorphic to the Clifford algebra $C(1, 1)$ [17, 18, 19]:

$$[\Gamma, \Gamma]_+ = \hbar \text{id}, \quad [\Upsilon, \Upsilon]_+ = -\hbar \text{id}, \quad [\Gamma, \Upsilon]_+ = 0. \quad (24)$$

The key point here is the symmetry of the super Poisson bracket.

The conclusion is that, in their quantum version, odd variables can give rise to meaningful observables. The classical limit $\hbar \rightarrow 0$ leaves the superspace mathematical structure, but it does not produce classical, fermionic observables.

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A Some basic definitions

Definition A.1. *Extension of scalars.* Let \mathcal{A} and \mathcal{A}' be two commutative superalgebras over k and let $f : \mathcal{A} \rightarrow \mathcal{A}'$ be a morphism of superalgebras. Then \mathcal{A}' is an \mathcal{A} -module with action

$$\begin{aligned}\mathcal{A} \otimes \mathcal{A}' &\longrightarrow \mathcal{A}' \\ a \otimes a' &\longrightarrow f(a)a' .\end{aligned}$$

If $M_{\mathcal{A}}$ is an \mathcal{A} -module we can define an \mathcal{A}' -module as the tensor product

$$M_{\mathcal{A}'} := \mathcal{A}' \otimes_{\mathcal{A}} M_{\mathcal{A}} = \mathcal{A}' \otimes M_{\mathcal{A}} / \langle a' \otimes a \cdot m - a' f(a) \otimes m \rangle$$

with action

$$\begin{aligned}\mathcal{A}' \otimes M_{\mathcal{A}'} &\longrightarrow M_{\mathcal{A}'} \\ b' \otimes [a' \otimes m] &\longrightarrow [(b'a') \otimes m] .\end{aligned}$$

We say that $M_{\mathcal{A}'}$ is the extension of scalars of $M_{\mathcal{A}}$ to \mathcal{A}' . □

Definition A.2. *Sheaf over a topological space.* Let $|X|$ be a topological space. A *presheaf* on $|X|$ assigns to each open set $U \subset |X|$ a set $\mathcal{F}(U)$ (it can be an abelian group, an algebra, a superalgebra, a module, ...) and to every pair of open sets $U \subset V \subset |X|$ a *restriction map*

$$\mathcal{F}(V) \xrightarrow{\text{res}_{V,U}} \mathcal{F}(U)$$

satisfying

1. $\text{res}_{U,U} = \text{id}$.
2. $\text{res}_{V,U} \circ \text{res}_{W,V} = \text{res}_{W,U}$ for all $U \subset V \subset W \subset |X|$.

The elements of $\mathcal{F}(U)$ are called *local sections* of \mathcal{F} over U and the elements of $\mathcal{F}(|X|)$ are called *global sections*.

A presheaf is a *sheaf* if it satisfies the condition that, for each open covering $\{U_\alpha\}_{\alpha \in A}$ of an open set U (in particular, of the total space $|X|$), and each collection of elements $\{f_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha \in A}$ such that

$$\text{res}_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = \text{res}_{U_\beta, U_\alpha \cap U_\beta}(f_\beta), \quad \forall \alpha, \beta \in A,$$

there exists a unique element $f \in \mathcal{F}(U)$ such that

$$\text{res}_{U, U_\alpha}(f) = f_\alpha, \quad \forall \alpha \in A.$$

□

Example A.3.

1. Continuous, differentiable, real analytic or complex analytic functions on a topological space are all sheaves of algebras.
2. Sections of a vector bundle over a topological space are a sheaf of modules over some algebra of functions.
3. Constant functions over a topological space are, generically, only a presheaf. If the space is connected, then the sheaf condition is satisfied. Also, on a not necessarily connected space, one can define the sheaf of *locally constant functions*, that is, functions that are constant on an open neighborhood of each point.

□

Definition A.4. *Stalk of a sheaf over a point.* Let \mathcal{F} be a sheaf of abelian groups (all the sheaves that we use are so) over the topological space $|X|$. Let $x \in |X|$. The *stalk of \mathcal{F} at x* , denoted as \mathcal{F}_x is the direct limit (see for example Ref. [6]) of the family of abelian groups $\mathcal{F}(U)$ running over all neighborhoods U of $x \in |X|$.

□

Definition A.5. A *morphism of sheaves* $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ over the same topological space $|X|$ is a collection of maps $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$, $U \subset_{\text{open}} |X|$ such that for every pair of open sets $V \subset U \subset |X|$ the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \\ \text{res}_{V,U} \downarrow & & \downarrow \text{res}_{V,U} \\ \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \end{array}$$

commutes

□

B Notation for spinors

Let θ^α with $\alpha = 1, 2$ odd coordinates in some affine superspace. As customary, we define

$$(\epsilon_{\alpha\beta}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\epsilon^{\alpha\beta}) := \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} \epsilon_{\beta\gamma} = \delta_\gamma^\alpha,$$

where sum over repeated indices is understood. We also define

$$\theta_\alpha := \epsilon_{\alpha\beta} \theta^\beta, \quad \theta^\alpha = \epsilon^{\alpha\beta} \theta_\beta.$$

In general, for any two pairs of odd quantities, θ^α and ψ^α , anticommuting among them

$$\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha, \quad \psi^\alpha \psi^\beta = -\psi^\beta \psi^\alpha, \quad \theta^\alpha \psi^\beta = -\psi^\beta \theta^\alpha,$$

one has

$$\theta^\alpha \psi_\alpha = \psi^\alpha \theta_\alpha, \quad \theta^\alpha \theta_\alpha = 2\theta^1 \theta^2,$$

and the so called *Fierz identities*

$$\begin{aligned} \theta^\alpha \theta^\beta &= -\epsilon^{\alpha\beta} \theta^1 \theta^2 = -\frac{1}{2} \epsilon^{\alpha\beta} \theta^\gamma \theta_\gamma, \\ \theta_\alpha \theta_\beta &= \epsilon_{\alpha\beta} \theta_1 \theta_2 = \frac{1}{2} \epsilon_{\alpha\beta} \theta^\gamma \theta_\gamma, \\ (\psi_1 \psi_2) \psi_3 + (\psi_3 \psi_1) \psi_2 + (\psi_2 \psi_3) \psi_1 &= 0. \end{aligned} \tag{25}$$

The Pauli matrices are

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and they satisfy the relations

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k, \quad i, j, k = 1, 2, 3,$$

where, as usual, ϵ_{ijk} is the totally antisymmetric tensor with $\epsilon_{123} = 1$.

C Supersymmetry transformations.

We give here the relations of the supertranslation algebra acting on the Minkowski superspace. We follow the conventions of Ref. [20]. A basis of the supertranslation Lie algebra is given by

$$\begin{array}{lll} Q_\alpha, & \bar{Q}_{\dot{\alpha}}, & \alpha, \dot{\alpha} = 1, 2 \quad (\text{odd}), \\ P_\mu, & & \mu = 0, \dots, 3 \quad (\text{even}). \end{array} \quad (26)$$

The standard real form makes P_μ real and $\bar{Q}_{\dot{\alpha}}$ the complex conjugate of Q_α . The commutation relations among the generators (26) are

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma_\mu)_{\alpha\dot{\beta}} P^\mu,$$

and the rest zero (the Pauli matrices are listed in Appendix B). The action of the supertranslation algebra on chiral superfields is as follows: Let ξ^α , $\bar{\xi}^{\dot{\alpha}}$ denote the odd supertranslation parameters and a^μ the even ones. The infinitesimal transformations on the component fields A , ψ_α and F

$$\begin{aligned} \delta_a(\cdot) &= a^\mu i \partial_\mu(\cdot) \quad (\text{applied to } A, \psi_\alpha \text{ and } F), \\ \delta_\xi A &= \xi^\alpha \psi_\alpha, \\ \delta_\xi \psi_\alpha &= 2(\sigma^\mu)_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}} i \partial_\mu A + 2F \xi_\alpha, \\ \delta_\xi F &= -i \partial_\mu \psi^\alpha (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}}. \end{aligned} \quad (27)$$

Notice that acting with $\bar{Q}_{\dot{\alpha}}$ on a chiral superfield gives a non chiral superfield, so only P_μ and Q_α have a well defined action on the set of chiral superfields.

References

- [1] P. Deligne, J.W. Morgan *Notes on Supersymmetry (following J. Bernstein)* in *Quantum fields and strings: a course for mathematicians*. (P. Deligne, P. Etingof, D. S. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D.R. Morrison and E. Witten, eds.) Vol 1. American Mathematical Society, Providence, (1999).
- [2] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction*. Courant Lecture Notes, 1. AMS (2004).
- [3] C. Carmeli, L. Caston and R. Fiorese *Mathematical Foundations of Supersymmetry*, with an appendix with I. Dimitrov, EMS Serie soif Lectures in Mathematics. European Mathematical Society, Zurich (2011).
- [4] R. Fiorese, M. A. Lledó, *The Minkowski and conformal superspaces: the classical and the quantum pictures*, World Scientific Publishing, Singapore, (2015).
- [5] A. S. Shvarts, *On the definition of superspace*. Theoretical and Mathematical Physics **60** n 1 (1984) 657660.
- [6] D. Eisenbud, J. Harris, *The geometry of schemes*, Springer (2000).
- [7] D.V. Volkov, V.P. Akulov, *Is the Neutrino a Goldstone Particle?* Physics Letters B **46** (1973) 109-110.
- [8] M. Rocek, *Linearizing the Volkov-Akulov Model*. Physical Review Letters **41** (1978) 451-453.
- [9] R. Casalbuoni, D. De Curtis, D. Dominici, F. Feruglio, R. Gatto, *Non-Linear realization of supersymmetry algebra from supersymmetric constraint*. Physics Letters B **220** n.4. (1989) 569-575.
- [10] Z. Komargodski, N. Seiberg, *From linear SUSY to constrained superfields*. JHEP 0909 (2009) 066.
- [11] D. M. Ghilencea, *Comments on the nilpotent constraint on the goldstino superfield*. Modern Physics Letters A **31** no.12 (2016) 1630011.
- [12] A. Brignole, F. Feruglio, F. Zwirner, *On the effective interactions of a light gravitino with matter fermions*. JHEP 11 (1997) 001.

- [13] G. Dall'Agata, S. Ferrara, F. Zwirner *Minimal scalar-less matter-coupled supergravity*. Physics Letters B **752** (2016) 263-266.
- [14] B. Vercnocke, Timm Wrase *Constrained superfields from an anti-D3-brane in KKLT*. Journal of High Energy Physics **1608** (2016) 132.
- [15] R. Kallosh, B. Vercnocke, Timm Wrase, *String Theory Origin of Constrained Multiplets*. Journal of High Energy Physics **1609** (2016) 063.
- [16] R. Fioresi, M. A. Lledó, V. S. Varadarajan. *The Minkowski and conformal superspaces*, Journal of Mathematical Physics, **48** (2007) 113505.
- [17] F. A. Berezin, *The method of second quantization*. Academic Press, (1996); F. A. Berezin and M. S. Marinov, *Particle spin dynamics as the Grassmann variant of classical mechanics*. Annals of Physics **104** (1997) 336-362 .
- [18] S. Ferrara, M. A. Lledó, *Some aspects of deformations of supersymmetric field theories*. Journal of High Energy Physics (2000) 05/008.
- [19] M. A. Lledó, *Deformed supersymmetric field theories*. Modern Physics Letters A **16** (2001) 305-310.
- [20] S. Ferrara, J. Wess, B. Zumino, *Supergauge multiplets and superfields*. Physics Letters B **51** (1974) 239-241.